[This question paper contains 4 printed pages]

09/5/18

Sr. No. of the Question Paper : **Roll No**__________

Unique Paper Code	:	217281
Name of the Paper	:	Chemistry (Credit Course-II) (CHCT -101)
Name of the Course	:	B.Sc. (H) Mathematics/ B.Sc. (G) Mathematical Science
Semester	:	II/IV
Duration	:	3 Hours
Maximum Marks	:	75 Marks

Instructions for Candidates

1. *Write your Roll No. on the top immediately on receipt of this question paper.*
2. Attempt 3 questions from **Section A** and 3 questions from **Section B.**
3. Indicate the section you are attempting by putting a heading and do not intermix the sections.
4. The questions should be numbered in accordance to the number in the question paper
5. Calculators and log tables may be used.

SECTION A

(Attempt three questions in all)

Q1 (a) Calculate the lattice energy of cesium chloride, CsCl using the following data :

Sublimation energy of cesium, Cs(s) (ΔH_{Sub}) = 79.9 $kJmol^{-1}$

1^{st} ionization energy for cesium Cs(g) (ΔH_{IE1}) = 374.05 $kJmol^{-1}$

Bond dissociation energy of chlorine gas, Cl_2(g) (ΔH_{BD}) = 241.84 $kJmol^{-1}$

Electron Affinity of chlorine gas Cl(g) (ΔH_{EA}) = -349 $kJmol^{-1}$

Enthalpy of formation of CsCl(s) (ΔH_f) = - 443 $kJmol^{-1}$ 4

(b) Can hypothetical cesium dichloride ($CsCl_2$) exist? Justify your answer. 2

(d) Write down the Born-Lande equation for the lattice energy of an ionic compound and define terms in it. 3

(d) Calculate the limiting radius ratio of cation to that of anion of an ionic compound when coordination number is six. Predict the geometry of BeS.

Given r_{Be}^{2+} = 59 pm and r_S^{2-} = 170 pm 3.5

Q2 (a) What are equivalent and non-equivalent hybrid orbitals? Using Bent rule, predict whether Cl-C-Cl angle is greater or smaller than tetrahedral angle (i.e. 109.5°) in CH_2Cl_2. Justify you answer. 4.5

(b) Which of the following compounds will have higher boiling point? Justify your answer.

i) HF or HCl

ii) *o*-nitrophenol or *p*-nitrophenol 2

(c) Sketch MO diagram for CO. Calculate its bond order. 3

(d) Predict the shapes of the following molecules using VSEPR theory:

$XeOF_4$, ClF_3, ICl_4^-

or

What type of hybridization is possible in the molecules CH_4, PF_5 and IF_7 ? 3

Q3 (a) What is the relationship between Δ_t and Δ_o? 1

(b) What is Jahn-Teller distortion? Cr(II) and Cu(II) coordination complexes show tetragonally distorted octahedral structures. Justify. 4

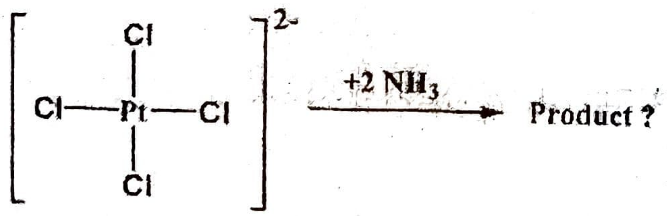
(c) Draw crystal field splitting diagram for *low spin* and *high spin* d^6 octahedral complexes. Calculate CFSE in terms of Δ (crystal field splitting energy) and P (pairing energy) for *low spin* d^6 octahedral complex. 4.5

(d) Which of the following complexes has higher value of Δ_o and why ?

i) $[Co(H_2O)_6]^{3+}$ or $[Co(H_2O)_6]^{2+}$

ii) $[Co(NH_3)_6]^{3+}$ or $[CoF_6]^{3+}$ 3

Q4 (a) What is trans-effect? Predict the product in the following reaction: 3



(b) Explain electrostatic polarization theory and π-bonding theory of trans-effect. 3.5

(c) Suggest a mechanism for direct electron transfer from $[Fe(CN)_6]^{4-}$ to $[Fe(CN)_6]^{3-}$

or

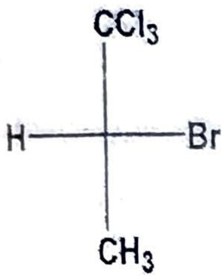
The reduction of $[Co(NH_3)_5Cl]^{2+}$ is about 10^{10} times faster than the reduction 3 of $[Co(NH_3)_6]^{3+}$. Name and discuss the mechanism involved.

(d) Sketch the crystal field splitting in a square planar complex. 3

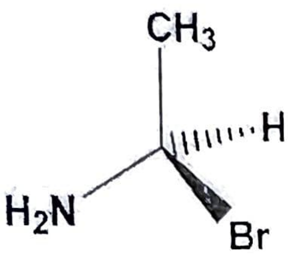
SECTION B

(Attempt three questions in all)

Q5 (a) Using sequence rules, assign R/S notations to each of the stereocentres in the following configurations: 4



(i)



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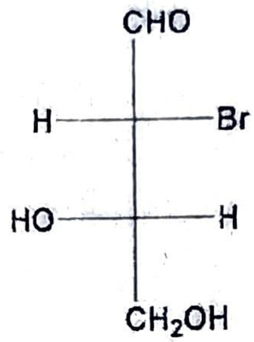
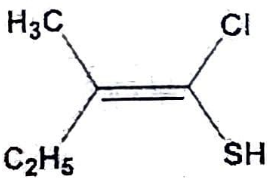


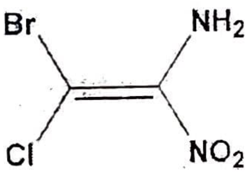
(iii)

(b) Draw the Newman and Sawhorse projections for anti, gauche, eclipsed and fully eclipsed conformations of 1,2-dichloroethane specifying the dihedral angle between two chloro substituents and also indicate which conformation is more stable and why ? 6.5

(c) Assign E and Z notations to the following olefins and write the steps involved



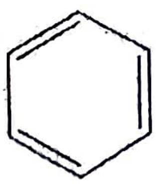
(i)



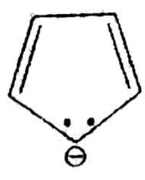
(ii) 2

Q6 (a) Explain, why the chair conformation of cyclohexane is more stable than its boat conformation with the help of Newman projections. 2

(b) Which of the following cyclic organic compounds are aromatic and why?



(i)



(ii)

(iii) 4.5

(c) Explain why (any three)?

i) Cyclohexylamine is more basic than aniline.

ii) Rate of nitration of chlorobenzene is greater than rate of chlorination of nitrobenzene.

iii) Allyl free radical is more stable.

iv) Chloroacetic acid is stronger acid than acetic acid. 3 x 2 =

Q7 (a) Write the products, name of the reaction and outline the mechanism of following reaction:

$$2CH_3CHO \xrightarrow[\text{NaOH}]{40\%} A + B$$ 4

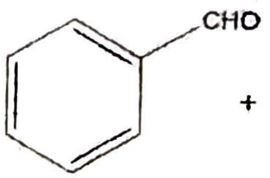
(b) Give the mechanism of Claisen condensation. 3

(c) What happens when methyl magnesium bromide reacts with ethanal (CH_3CHO) followed by hydrolysis? 2

(d) Why tertiary haloalkanes undergo nucleophile substitution reaction via S_N1 mechanism? 2

(e) Explain Saytzeff's rule with suitable example. 1.5

Q8 (a) Complete the following reaction and indicate the name reaction involved and write the mechanism of the reaction 4



$$+ (CH_3CO)_2O \xrightarrow[\text{Heat}]{CH_3COONa} ?$$

(b) Write the product and classify the following reactions as addition, elimination or substitution reaction. 4.5

$$CH_3COCH_3 + HCN \longrightarrow ?$$

$$CH_3CH_3CH_2Br + \text{ethanolic KOH} \longrightarrow ?$$

$$CH_3CH_2Cl + NaCN \longrightarrow ?$$

(c) Write a short note on addition or condensation polymerization. Also draw the monomeric unit(s) present in natural rubber (polymer) indicating its name. 4

(31)

13/5/18

SP AC MQP: 5716

Unique Paper Code : 235201

Name of Paper : Differential Equations and Mathematical Modeling –I (MAHT-201)

Name of Course : B.Sc. (H) Mathematics

Semester : II

Duration : 3 hours

Maximum Marks : 75 Marks

Instructions for Candidates:

1. Attempt all questions.

2. Use of non-programmable scientific calculator is allowed.

Section-I

Q1. *Attempt any three parts of the following:* (5+5+5)

(a) Solve the differential equation

$$y^2(xy' + y)\sqrt{1 + x^4} = x.$$

(b) Solve the differential equation $yy'' + (y')^2 = 0$, in which the independent variable x is missing. ??

(c) Determine the most general function *M(x, y)* such that the equation $M(x, y)dx + (2x^2y^3 + x^4y)dy = 0$ is exact and hence solve it.

(d) Solve the initial value problem $\frac{dr}{d\theta} + r\, tan\theta = cos^2\theta,\ r(\pi/_4) = 1.$

Q2. *Attempt any two parts of the following:* (5+5)

(a) An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $\frac{v^2}{800}\ ft/s^2$ due to air resistance. How high in the air does it go?

(b) At time t=0 the bottom plug (at the vertex) of a full conical water tank 16 ft high is removed. After 1 hour the water in the tank is 9 ft deep. When will the tank be empty?

(c) There are now about 3300 different human "language families" in the whole world. Assume that all these are derived from a single original language, and that a language family develops into 1.5 language families every 6000 years. About how long ago was the original human language spoken?

Section-II

Q3. *Attempt any two parts of the following:* (7.5+7.5)

(a) Alcohol is unusual in that it is removed from the bloodstream by a constant amount each time period, independent of the amount in the bloodstream. This removal can be modeled by a Michaelis-Menten type function $y' = \frac{-k_3 y}{(y+M)}$ where y(t) is the amount (BAL) of alcohol in the bloodstream at time t, k_3 is a positive constant and M a small positive constant.

(i) If y is large compared with M then show that $y' \cong -k_3$. Solve for y in this case.

(ii) Alternatively, as y decreases and becomes small compared with M, show that then $y' \cong \frac{-k_3 y}{M}$. Solve for y in this case.

(iii) Give a rough sketch of the solution function for $y' = \frac{-k_3 y}{(y+M)}$ assuming that, initially, y is much greater than M. Indicate clearly how the graph changes in character when y is small compared with M, compared with when y is large compared with M. Show how the solution behaves as $t \to \infty$.

(iv) When and why would this function be more suitable than simply using $y' = -k_3$ to model the removal rate?

(b) In view of potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of population at the current time. One harvesting model that takes this into account is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h_0 X.$$

(i) How many equilibrium populations are there? Find them.

(ii) At what critical harvesting rate can extinction occur using this model?

(c) Consider the population of a country. Assume constant per-capita birth and death rates, and that the population follows an exponential growth (or decay) process. Assume there to be a significant immigration and emigration of people into and out of the country.

(i) Assuming the overall immigration and emigration rates are constant, formulate a single differential equation to describe the population size over time.

(ii) Suppose instead that all immigration and emigration occurs with a neighboring country, such that the net movement from one country to other is proportional to the population difference between the two countries and such that people move to the country with the larger population. Formulate a coupled system of equations as a model for this situation.

In both (i) and (ii) start with appropriate word equations and ensure all variables are defined. Give clear explanations of how the differential equations are obtained from the word equations.

Section-III

Q4. *Attempt any four parts of the following:* (5+5+5+5)

(a) Solve the Euler equation

$$x^2y'' + xy' + 9y = 0.$$

(b) State and prove the principle of superposition for homogeneous linear differential equations of second order. Can linearity be dropped? Justify your answer.

(c) Use the method of variation of parameters to find a particular solution of $y'' + 4y = \sin^2 x$.

(d) Use the method of undetermined coefficients to find a particular solution of $y''' + y'' = 3e^x + 4x^2$.

(e) A body with mass 250 g is attached to the end of a spring that is stretched 25 cm by a force of 9 N. At time t=0, the body is pulled 1 m to the right, stretching the spring and is set in motion with an initial velocity of 5 m/s to the left. Find x(t) in the form $C\cos(w_o t - \alpha)$ and find the amplitude of the motion of the body. ?

Section-IV

Q5. *Attempt any two parts of the following:* (7.5+7.5)

(a) In a long range battle, neither army can see the other, but fires into a given area. A simple mathematical model describing this battle is given by the coupled differential equations

$$\frac{dR}{dt} = -c_1RB\,, \qquad \frac{dB}{dt} = -c_2RB$$

where c_1 and c_2 are positive constants.

(i) Use the chain rule to find a relationship between R and B, given the initial number of soldiers for the two armies are r_0 and b_0 respectively.

(ii) Draw a sketch of typical phase-plane trajectories.

(iii) Explain how to estimate the parameter c_1 given that the blue army fires into a region of area A.

(b) A model of a three species interaction is

$$\frac{dX}{dt} = a_1X - b_1XY - c_1XZ, \quad \frac{dY}{dt} = a_2XY - b_2Y, \quad \frac{dZ}{dt} = a_3XZ - b_3Z,$$

where a_i, b_i, c_i, for i=1,2,3, are all positive constants. Here $X(t)$ is the prey density and $Y(t)$ and $Z(t)$ are the two predator species densities.

(i) Find all possible equilibrium populations.

(ii) Is it possible for all three populations to coexist in equilibrium?

(iii) What does this suggest about introducing an additional predator into ar ecosystem?

(c) Consider a population split into two groups: adults and juveniles, where the adults give birth to juveniles but juveniles are not yet fertile. Eventually juveniles mature into adults. Assume constant per capita birth and death rates for the population and that the young mature into adults at a constant per capita rate σ.
Starting from a compartmental diagram and suitable word equations formulate a pair of differential equations describing the density of adults and density of juveniles at any time. Define all variables and parameters used.

(22) 15/5/2018

[This question paper contains 2 printed pages]

Your Roll No........................

Sr. No. of Question Paper : 5717

Unique Paper Code : 235203

Name of the Course : B.Sc. (Hons) Mathematics H

Name of the Paper : MAHT-202 : Analysis-II

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **three** parts from each question.
3. **All** questions are compulsory.

1.(a) Use the ϵ-δ definition of the limit, find $\lim_{x\to 1} f(x)$ where $f(x) = \frac{x}{1+x}$. 5

(b) State and prove Sequential Criterion for Limits. 5

(c) State Squeeze Theorem. Show that $\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$. 5

(d) Let $f(x) = |x|^{-1/2}$ for $x \neq 0$. Show that $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = +\infty$. 5

2.(a) Let $A \subseteq R$, let $f: A \to R$, and let $f(x) \geq 0$ for all $x \in A$. Prove that if f is continuous on A, then $\sqrt{f}$ is continuous on A. 5

(b) Determine the points of continuity of the function $x - [x]$, where $x \to [x]$ denotes the greatest integer function. 5

(c) Let $I = [a, b]$ and let $f : I \to R$ and $g : I \to R$ be continuous functions on I. Show that the set $E = \{x \in I : f(x) = g(x)\}$ has the property that if $(x_n) \subseteq E$ and $x_n \to x_0$, then $x_0 \in E$. 5

(d) State Bolzano's Intermediate Value theorem. Let f and g be continuous functions on [a, b] such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Show that $f(c) = g(c)$ for at least one c in [a, b]. 5

3.(a) Define uniformly continuous function on a set $A \subseteq R$. Prove that if f and g are each uniformly continuous on R, then the composite function $f \circ g$ is uniformly continuous on R. 5

(b) State non- uniform continuity criteria. Show that the function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$. 5

(c) Determine where the function $f(x) = |x| + |x+1|$, $x \in R$ is differentiable and find the derivative. 5

delete

(d) Suppose that $f : R \to R$ is differentiable at c and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$. 5

4.(a) Let $f : I \to R$ be differentiable on the interval I. Show that f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$. 5

(b) Let $f : I \to R$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Prove that either the derivative of f at c does not exist or it is equal to zero. 5

(c) State the Mean Value Theorem. Use the theorem to prove that

$$\frac{x-1}{x} < \ln x < x - 1, \text{ for } x > 1.$$ 5

(d) Suppose that $f : [0,2] \to R$ is continuous on $[0,2]$ and differentiable on $]0,2[$ and that $f(0) = 0, f(1) = 1, f(2) = 1$.

(i) Show that there exists $c_1 \in]0,1[$ such that $f'(c_1) = 1$.
(ii) Show that there exists $c_2 \in]1,2[$ such that $f'(c_2) = 0$.
(iii) Show that there exists $c \in]0,2[$ such that $f'(c) = \frac{1}{7}$. 5

5.(a) State and prove Taylor's Theorem. 5

(b) Obtain Maclaurin's series expansion for the function $\sin x$. 5

(c) Show that $1 - \frac{1}{2}x^2 \leq \cos x$ for all $x \in R$. 5

(d) Define a convex function on an interval $I \subseteq R$. Check which of the following functions are convex :

(i) $\cot x, \ x \in \left[0, \frac{\pi}{2}\right]$
(ii) $|x-1|, \ x \in [0,2]$ 5

(93)

Sr. No. of Question Paper : 5718

Unique Paper Code : 235204

Name of the Paper : Probability and Statistics - MAHT 203

Name of the Course : B.Sc.(H) Mathematics

Semester : II

Duration : 3 Hours

Maximum Marks : 75

<u>Instructions for Candidates</u>

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. In all there are **six** questions.
3. Question No. 1 is compulsory and it contains **five** parts of **3** marks each.
4. In Question No. **2** to **6**, attempt any **two** parts from **three** parts. Each part carries **6** marks.
5. Use of scientific calculator is allowed.

SET-B

Q1 (i) If C_1 and C_2 are events in a sample space S, then prove that

$$P(C_1 \cap C_2) \geq P(C_1) + P(C_2) - 1.$$

(ii) If X and Y are independent random variables then show that $Cov\,(X,Y) = 0$.

(iii) If X has a Poisson distribution with $P(X = 1) = P(X = 2)$, what is $P(X = 1 \; or \; 2)$?

(iv) Let X have the pdf

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0 & , elsewhere \end{cases}$$

Find $E\big(X(1-X)\big)$.

(v) Find the value of c for which the two random variables X and Y have the joint pmf

$$p(x,y) = c(x^2 + y^2)\,, \text{ for } x = -1, 0, 1, 3\,;\; y = -1, 2, 3\,, \text{ zero elsewhere.}$$

Q2 (a) Let $\{C_n\}$ be an increasing sequence of events. Then prove that

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P\left(\bigcup_{n=1}^{\infty} C_n\right).$$

(b) (i) There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered 1,2,3,4,5 respectively and the blue chips are numbered 1,2,3 respectively. If 2 chips are to be drawn at random and without replacement, find the probability that these chips have either the same number or the same colour.

(b) (ii) Given the cdf

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x+2}{4}, & -1 \leq x < 1 \\ 1, & 1 \leq x. \end{cases}$$

Find $P\left(-\frac{1}{2} < X \leq \frac{1}{2}\right)$, $P(X = 1)$ and $P(2 < X \leq 3)$.

(c) Find the moment -generating function of the geometric distribution. Hence or otherwise, find its mean and variance.

Q3 (a) Derive the following recursion formula for a random variable X having Poisson distribution with parameter λ :

$$\mu_{r+1} = \lambda\left[r\,\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right] \quad \text{for } r = 1,2,3, \ldots\ldots\ldots$$

where μ_r denotes the rth moment about mean.

Also find μ_2 and μ_3 using $\mu_0 = 1$ and $\mu_1 = 0$.

(b) (i) Show that if a random variable has an exponential density with the parameter θ , the probability that it will take on a value less than $-\theta . ln(1-p)$ is equal to p .

(ii) If a random variable has a uniform density with the parameters α and β, find its distribution function.

(c) Find the moment - generating function of the normal distribution. Hence or otherwise, find its mean and variance.

Q4 (a) Let X be a random variable having standard normal distribution, then show that :

(i) $\mu_r = 0$, when r is odd ;

(ii) $\mu_r = \dfrac{r!}{2^{r/2}\left(\frac{r}{2}\right)!}$, when r is even.

(b) Let $f(x,y) = 2,\ 0 < x < y,\ 0 < y < 1$, zero elsewhere be the joint pdf of X and Y . Show that the conditional means are, respectively, $\dfrac{1+x}{2},\ 0 < x < 1,$ and $\dfrac{y}{2},\ 0 < y < 1.$

(c) Let X and Y have the joint pmf described by the following table :

(x, y)	(0,0)	(0,1)	(0,2)	(1,1)	(1,2)	(2,2)
$p(x,y)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{1}{12}$

Find the correlation coefficient of X and Y.

Q5 (a) Given the joint density

$$f(x,y) = \begin{cases} 6x, & 0<x<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the regression equation of X on Y.

(b) If $f(x,y) = e^{-x-y}$, $0 < x < \infty,\ 0 < y < \infty$, zero elsewhere, is the joint pdf of the random variables X and Y. Show that $M(t_1, t_2) = (1-t_1)^{-1}.(1-t_2)^{-1}$, $t_1 < 1$ and $t_2 < 1$. Hence show that $E\left(e^{t(X_1+X_2)}\right) = (1-t)^{-2},\ t < 1.$

(c) Suppose (X, Y) has a joint distribution with the variances of X and Y finite and positive. If $E(Y|X)$ is linear then show that $E(Y|X) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1)$.

Q6 (a) Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2. Transform the process into a Markov Chain and finds it transition probability matrix.

(b) State and prove Central Limit theorem for a sequence of independent and identically distributed random variables.

(c) Suppose we know that the number of items produced in a factory during a week is a random variable with mean 500.

(i) What can be said about the probability that this week's production will be atleast 1000 ?

(ii) If the variance of a week's production is know to equal 100, then what can be said about the probability that this week's production will be between 400 and 600 ?

(1)

This question paper contains 4 printed pages.

2018

Your Roll No. ....

S. No. of Paper : **6630** **HC**
Unique paper code : **32351201**
Name of the paper : **Real Analysis**
Name of course : **B.Sc. (Hons.) Mathematics**
Semester : **II**
Duration : **3 hours**
Maximum marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any ***three*** *parts from each question.*

All *questions are compulsory.*

1. (a) Prove that a lower bound v of a nonempty set S in **R** is the Infimum of S if and only if for every $\epsilon > 0$, there exists an $s_\epsilon \in S$ such that $s_\epsilon < v + \epsilon$.

(b) Let S be a nonempty bounded above set in **R**. Let $a > 0$ and $aS = \{as: s \in S\}$, then prove that Sup (aS) = a Sup S.

(c) If x and y are positive real numbers with $x < y$, then prove that there exists a rational number $r \in Q$ such that $x < r < y$.

(d) Show that $\text{Sup}\left\{1 - \frac{1}{n} : n \in \mathbf{N}\right\} = 1$. 5,5,5

P. T. O.

2. (a) Define limit point of a set in **R**. Prove that a point $c \in \mathbf{R}$ is a limit point of a set S if and only if every neighbourhood of c contains infinitely many points of S.

(b) Let (x_n) be a sequence of real numbers such that $\lim_{n\to\infty} x_n = x > 0$, then show that there exists a natural number K such that

$$\frac{x}{2} < x_n < 2x \qquad \forall\, n \geq K.$$

(c) Use the definition of limit to prove:

i. $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

ii. $\lim_{n\to\infty} \left(\frac{3n+2}{n+1}\right) = 3$.

(d) Let (x_n) be a sequence of positive real numbers such that $L = \lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n}\right)$ exists. Show that if $L < 1$, then (x_n) converges and $\lim_{n\to\infty} x_n = 0$. (5, 5, 5)

3. (a) Let (x_n) and (y_n) be sequences of real numbers such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then show that $\lim_{n\to\infty} x_n y_n = xy$.

(b) State Squeeze Theorem and hence prove that

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = b$$

where $0 < a < b$.

(c) State and prove Monotone Convergence Theorem.

(d) Let (x_n) be a sequence of real numbers defined by

$$x_1 = 8,\ x_{n+1} = \frac{x_n}{2} + 2 \quad \text{for } n \in \mathbf{N}.$$

Show that (x_n) is convergent and find its limit. (5, 5, 5)

4. (a) Show that the following sequences are divergent:

(i) $((-1)^n)$

(ii) $\left(\sin\left(\frac{n\pi}{3}\right)\right)$

(b) Define a Cauchy sequence and show that every Cauchy sequence of real numbers is bounded.

(c) Prove that the sequence (x_n), where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}, \qquad n \in N$$

is not a Cauchy sequence.

(d) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be infinite series of positive real numbers such that $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$. Show that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (5, 5, 5)

5. (a) State and prove *n*-th Root Test to test the convergence of an infinite series.

(b) Test for convergence any *two* of the following series:

i. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

ii. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

iii. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

(c) Define Conditional Convergence. Show that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)}$$

is conditionally convergent.

(d) Test the following series for Absolute convergence:

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{(n!)^2}{(2n)!}.$$

(5, 5, 5)

(2)

This question paper contains 5 printed pages. 29/5/18

Your Roll No. ....

HC

S. No. of Paper : **6631**

Unique Paper Code : **32351202**

Name of the Paper : **Differential Equations**

Name of the Course : **B.Sc. (Hons.) Mathematics – I**

Semester : **II**

Duration : **3 hours**

Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All the Sections are compulsory.
Use of non-programmable scientific calculator is allowed.

Section I

1. Attempt any *three* parts of the following: (5+5+5)

   a. Solve the differential equation:
   $$\left(2xe^{y}y^4+2xy^3+y\right)dx+\left(x^2e^{y}y^4-x^2y^2-3x\right)dy=0.$$

   b. Find the general solution of the differential equation:
   $$yy''+(y')^2=yy'.$$

   c. Solve the differential equation:
   $$\frac{dy}{dx}=\frac{x-y-1}{x+y+3}.$$

   d. Solve the initial value problem:
   $$\frac{dy}{dx}=2xy^2+3x^2y^2,\ y(1)=-1.$$

2. Attempt any *two* parts of the following: (5+5)

   a. A water tank has the shape obtained by revolving the parabola $x^2=by$ around the y axis. The water depth is 4 ft at 12 noon, when a circular plug in the bottom of the tank is removed. At 1 pm, the depth of the water is 1 ft. Find the depth $y(t)$ of water remaining after t hours. Also, find when the tank will be empty. If the initial radius of the top surface of the water is 2 ft, what is the radius of the circular hole in the bottom?

b. A certain piece of dubious information about phenyl ethyl amine in the drinking water began to spread one day in the city with a population of 100,000. Within a week 10,000 people heard this rumour. Assume that the rate of increase of the number who have heard the rumour is proportional to the number who have not heard it. How long will it be until half the population of the city has heard the rumour?

c. Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity υ, so that $d\upsilon/dt = -k\upsilon^2$. Show that :

$$\upsilon(t) = \frac{\upsilon_0}{1+\upsilon_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1+\upsilon_0 kt).$$

Section II

3. Attempt any *two* parts of the following: (8+8)

a. The following differential equation describes the level of pollution in the lake:

$$\frac{dC}{dt} = \frac{F}{V}(C_{in} - C)$$

where V is the volume, F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake. Let $V = 28\times10^6 m^3, F = 4\times10^6 m^3/month$. If only fresh water enters the lake,

i. How long would it take for the lake with pollution concentration $10^7\ parts/m^3$ to drop below the safety threshold $\left(4\times10^6\ parts/m^3\right)$?

ii. How long will it take to reduce the pollution level to 5% of its current level?

b. In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is:

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right)-h_0X.$$

i. Show that the only non-zero equilibrium population is :

$$X_e = K\left(1-\frac{h}{r}\right).$$

ii. At what critical harvesting rate can extinction occur?

c. In a simple battle model, suppose that soldiers from the red army are visible to the blue army, but soldiers from the blue army are hidden. Thus, all the red army can do is fire randomly into an area and hope they hit something. The blue army uses aimed fire.

i. Write down appropriate word equations describing the rate of change of the number of soldiers in each of the armies.

ii. By making appropriate assumptions, obtain two coupled differential equations describing this system.

iii. Write down a formula for the probability of a single bullet fired from a single red soldier wounding a blue soldier in terms of the total area A and the area exposed by a single blue soldier A_b .

iv. Hence write the rate of wounding of both armies in terms of the probability and the firing rate.

Section III

4. Attempt any *three* parts of the following: (6+6+6)

a. Find the general solution of the differential equation:

$$x^3 y''' + 6x^2 y'' + 4xy' = 0$$

b. Using the method of undetermined coefficients, solve the differential equation :

$$y''' - 2y'' + y' = 1 + xe^x, y(0) = y'(0) = y''(0) = 1 .$$

c. Using the method of variation of parameters, solve the differential equation :

$$y'' + 3y' + 2y = 4e^x .$$

d. Show that $y_1 = 1 \, and \, y_2 = \sqrt{x}$ are solutions of :

$$yy'' + (y')^2 = 0,$$

but the sum $y = y_1 + y_2$ is not a solution. Explain why.

Section IV

5. Attempt any *two* parts of the following: (8+8)

a. Consider a disease where all those infected remain contagious for life. A model describing this is given by the differential equations

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI$$

where β is a positive constant.

i. Use the chain rule to find a relation between S and I.

ii. Obtain and sketch the phase-plane curves. Determine the direction of travel along the trajectories.

iii. Using this model, is it possible for all the susceptible to be infected?

b. The predator-prey equations with additional deaths by DDT are:

$$\frac{dX}{dt} = \beta_1 X - c_1 XY - p_1 X, \frac{dY}{dt} = -\alpha_2 Y + c_2 XY - p_2 Y$$

where all parameters are positive constants.

i. Find all the equilibrium points.

ii. What effect does the DDT have on the non-zero equilibrium populations compared with the case when there is no DDT?

iii. Show that the predator fraction of the total average prey population is given by:

$$f = \frac{1}{1 + \left(\frac{c_1(\alpha_2 + p_2)}{c_2(\beta_1 - p_1)} \right)}$$

What happens to this proportion f as the DDT kill rates, p_1 and p_2, increase?

c. The pair of differential equations

$$\frac{dP}{dt} = rP - \gamma PT, \frac{dT}{dt} = qP,$$

where r, γ *and* q are positive constants, is a model for a population of microorganisms P, which produces toxins T which kill the microorganisms.

i. Given that initially there are no toxins and p_0 microorganisms, obtain an expression relating the population density and the amount of toxins.

ii. Hence, give a sketch of a typical phase-plane trajectory.

iii. Using phase-plane trajectory, describe what happens to the microorganisms over time.